Accelerated Convergent Motion Compensated Image Reconstruction

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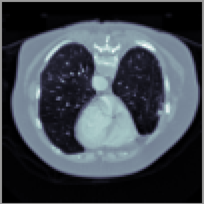
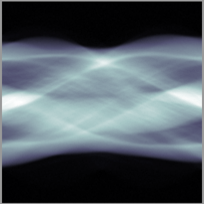
Motivation



(a) First motion state

(b) Last motion state



(c) Data

(d) Reconstruction

Figure: Motion in the subject introduces artefacts in the reconstruction

Goal

Proof of concept for a randomized algorithm performing Motion Compensated Image Reconstruction (MCIR) which...

- is faster than the non-randomized counterpart,
- is provenly convergent.

1. MCIR: framework
2. Proposed algorithm and theoretical results
3. Numerical experiments

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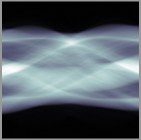
Framework

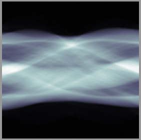
Data divided over N gates:

$$d_i \approx \underbrace{A}_{\text{forward op.}} \underbrace{D_i}_{\text{displacement op.}} x, \quad 1 \leq i \leq N.$$

$\xrightarrow{D_i}$

$\downarrow A$ $\quad \searrow AD_i$ $\quad \downarrow A$





Variational model

With Gaussian noise and l_2^2 regularizer:

$$\begin{aligned}
\min_x \quad O(x) \quad &:= \quad \frac{1}{N}\sum_{i=1}^{N} \|AD_i x - d_i\|^2 \quad + \quad \alpha\|x\|^2 \\
&:= \quad \underbrace{F(Lx)}_{\text{data fit}} \quad + \quad \underbrace{G(x)}_{\text{regularizer}} \ .
\end{aligned}$$

Convex setting.

Primal-Dual Hybrid Gradient (PDHG) algorithm

(also known as Chambolle-Pock algorithm)

Iterate

- $x^{k+1} = \text{prox}_{\tau G}(x^k - \tau L^* \bar{y}^k)$
- $y^{k+1} = \text{prox}_{\sigma F^*}(y^k + \sigma L x^k)$
- $\bar{y}^{k+1} = y^{k+1} + \theta_k (y^{k+1} - y^k)$

$\rightarrow$ convergent algorithm

$\rightarrow$ each iteration requires the evaluation of $L = (AD_1, \dots AD_N)$ and L^*: the computational cost scales linearly with the number of gates.

1 MCIR: framework

2 Proposed algorithm and theoretical results

3 Numerical experiments

Randomized algorithm

- Idea use only one gate, picked at random, for each iteration.
- How? use Stochastic Primal-Dual Hybrid Gradient (SPDHG) algorithm [Chambolle, Ehrhardt, Richtárik, Schönlieb, 2018]

$$\begin{aligned} \min_x \quad O(x) \quad &:= \quad \tfrac{1}{N}\textstyle\sum_{i=1}^{N} \|AD_i x - d_i\|^2 \quad + \quad \alpha\|x\|^2 \\ &= \quad \textstyle\sum_{i=1}^{N} F_i(L_i x) \quad + \quad G(x). \end{aligned}$$

Randomized algorithm

- Idea use only one gate, picked at random, for each iteration.
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Convex setting

SPDHG

Iterate

- $x^{k+1} = \text{prox}_{\tau G}(x - \bar{z}^k)$
- Pick a gate i with probability p_i
- $y_i^{k+1} = \text{prox}_{\sigma_i F_i^*}(y_i^k + \sigma_i L_i x^k)$ and $y_j^{k+1} = y_j^k$ for $j \neq i$
- $\delta^k = L_i^*(y_i^{k+1} - y_i^k)$
- $\bar{z}^{k+1} = \bar{z}^k + (1 + \theta_k p_i^{-1})\delta^k$

SPDHG

Iterate

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- Pick a gate i with probability p_i
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$\rightarrow$ convergent algorithm

$\rightarrow$ each iteration requires the evaluation of only one $L_i = AD_i$ and L_i^*: the computational cost scales constantly with the number of gates.

Theoretical rates of convergence

In the strongly convex - strongly smooth setting, PDHG and SPDHG converge linearly with known optimal per epoch rates [Chambolle et al., 2011], [Chambolle et al., 2018]:

$$\|x^K_{\text{PDHG}} - x^*\|^2 \leq C(r_N^{\text{PDHG}})^K$$
$$\mathbb{E}\left[\|x^K_{\text{SPDHG}} - x^*\|^2\right] \leq \tilde{C}(r_N^{\text{SPDHG}})^K.$$

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$$\mathbb{E}\left[\|x^K_{\text{SPDHG}} - x^*\|^2\right] \leq \tilde{C}(r_N^{\text{SPDHG}})^K.$$

Theorem

For N gates and well-chosen step-sizes, it stands that:

$$r_N^{PDHG} = 1 - \frac{2}{1 + \sqrt{1 + \frac{1}{\alpha N}\|(L_1, \dots, L_N)\|^2}},$$

$$r_N^{SPDHG} = \left(1 - \frac{2}{N\left(1 + \sqrt{1 + \frac{1}{\alpha N}\max_i \|L_i\|^2}\right)}\right)^N.$$

Theoretical rates of convergence

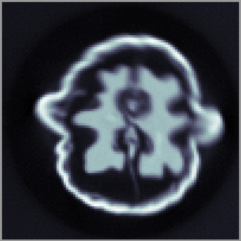
For a moderately conditioned problem such that $\kappa = \frac{\|A\|^2}{\alpha} \geq 16$,

$$r_N^{\mathrm{SPDHG}} \approx \left(1 - \frac{2}{N\left(1 + \sqrt{1 + \frac{\kappa}{N}}\right)}\right)^N \quad < \quad 1 - \frac{2}{1 + \sqrt{1 + \kappa}} \approx r_N^{\mathrm{PDHG}}.$$

Numerical application: rigid motion ($N = 20$ gates)

(a) First motion state

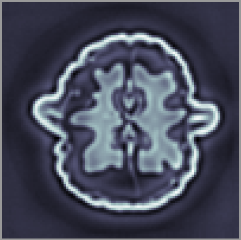
(b) Last motion state



(c) Converged no-MC

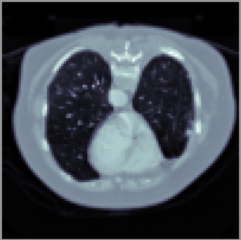
(d) Converged MC

(e) MC SPDHG after 30 epochs



(f) MC PDHG after 30 epochs

Numerical application: non-rigid motion ($N = 10$ gates)


(a) First motion state


(b) Last motion state

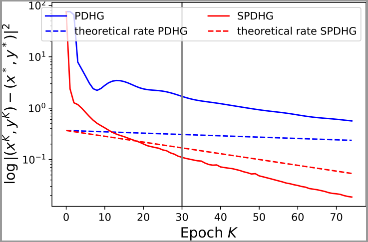

(c) Converged no-MC


(d) Converged MC

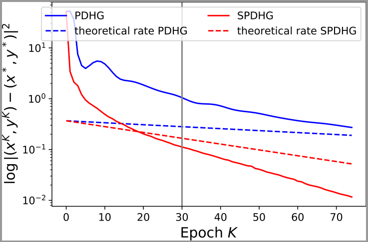

(e) MC SPDHG after 30 epochs


(f) MC PDHG after 30 epochs

Numerical application: convergence rates



(a) Rigid motion



(b) Non-rigid motion

Figure: SPDHG's linear convergence is faster than PDHG's

Contributions

We proposed a randomized algorithm for Motion Compensated Image Reconstruction with the following characteristics . . .

- $\rightarrow$ is provenly convergent,
- $\rightarrow$ requires the same computational effort than the non-motion compensated reconstruction per iteration,
- $\rightarrow$ [in proof-of-concept setting]
  - $\rightarrow$ a theoretical speed-up is proved on linear rates
  - $\rightarrow$ a practical speed-up is observed on synthetic experiments.